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# Jaynes–Cummings Model and Trapping of Atoms

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We present a semiclassical approach to the trapping problem of atoms in the Jaynes–Cummings model. We express detuning in two ways, first by considering detuning as the superposition of many harmonic waves, ther other by establishing a relationship between detuning and the distribution of the momentum of atoms. In these two cases we find the appearance of trapping atoms; we discuss the quantum chaotic movement of atoms irradiated with a strong laser.

## 1. INTRODUCTION

During the last two decades, ion trapping has attracted increasing theoretical and experimental attention. There are many different quantum models to explain the trapping of ions or atoms, such as the MOT or VSCPT models (Aspect *et al.*, 1988; Barenco *et al.*, 1995; Cirac and Zoller, 1995; Gupta *et al.*, 1996; Kasevich and Chu, 1992; Lee *et al.*, 1996; Leopold and Percival, 1978; Raab *et al.*, 1987). The Jaynes–Cummings (JC) model is an important full quantum mechanical model of two-level atoms interacting with a laser. We have discussed a unified and standardized procedure to solve full quantum nonlinear JC models (Yang *et al.*, 1997). In this paper, we will present a semiclassical method to study the trapping of atomis in JC models and their quantum chaotic motion. Leopold and Percival (1978) used classical chaos theory to describe successfully the interaction between a high-excited-state atom and microwaves. Can we use the JC model for the interaction between atoms and many photons according to the same method?

We begin with the following system:

$$H_q = \omega b^\dagger b + \frac{1}{2} E_{12}\sigma_z + H_{\text{int}} \tag{1}$$

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where $b^\dagger$ and $b$ are the creation and annihilation operators of the photon, respectively, $\omega$ is the photon energy ($\hbar = 1$), $E_{12}$ is the difference between two levels of the atom, $\sigma_z$ is the transition operator, $H_{\text{int}} = g(b^+\sigma_{-+} + b\sigma_{+-})$ is the interaction Hamiltonian, and $g$ is the coupling coefficient. Here $\sigma_{-+} = |-\rangle\langle+|$B and $\sigma_{+-} = |+\rangle\langle-|$. We can express (1) formally as (Yang *et al.*, 1997)

$$H_q' = \omega(N - 1) + \frac{1}{2}\overline{\Omega}\sigma_{\overline{x}} \tag{2}$$

where

$$\sigma_{\overline{x}} = \frac{\Delta}{\overline{\Omega}}\sigma_z + \frac{\Omega}{\overline{\Omega}}\sigma_x, \qquad N = b^\dagger b + \sigma_{++}, \qquad \overline{\Omega} = \sqrt{\Delta^2 + \Omega^2}$$

and

$$\Omega = 2g\sqrt{b^\dagger b + \sigma_{++}}$$

$N$ is the sum of the photon number and the number of the atomic excited state, and is a conserved quantity; $\Omega$ is the Rabi operator, and $\sigma_x$, $\sigma_z$ are Pauli operators. Equation (2) describes a spin-1/2 linear harmonic oscillator in a magnetic field. Therefore a two-level atom interacting with photons is equivalent to a linear harmonic oscillator interacting with a magnetic field.

When the $H_{\text{int}}' = 1/2\ \overline{\Omega}\sigma_{\overline{x}}$ term of equation (2) is very large, it is very difficult to apply quantum perturbation theory. We give the classical correspondence to equation (2) as follows:

$$H_c = \frac{1}{2}\dot{z}^2 + \frac{1}{2}\omega^2 z^2 + \frac{1}{2}\Delta \tag{3}$$

The first two terms correspond to a linear harmonic oscillator with frequency $\omega$, the third term is the average value in the $x$ and $y$ direction of $H_{\text{int}}'$; $\langle\sigma_x\rangle = 0$. We obtain

$$H_{\text{int}}' = \frac{1}{2}\overline{\Omega}\sigma_z\Delta\Big/\overline{\Omega} = \pm\Delta/2 \tag{4}$$

We have absorbed the sign into $\Delta$. Equation is the basis for studying the trapping problem of atoms in the JC model.

## 2. TWO METHODS TO CONSTRUCT DETUNING

in order to solve equation (3), we have to know the concrete form of the detuning. There are two ways to do this; one is to consider detuning as the superposition many harmonic waves, the other is to establish a relationship between detuning and the distribution of the momentum of atoms. We illustrate the two methods using two examples.

**1.** $\Delta$ expanded as the superposition of many harmonic waves. Let

$$\Delta = A \cos qx \sum_{n=-\infty}^{\infty} \cos(n \frac{2\pi t}{T} \tag{5}$$

Because of

$$\sum_{n=-\infty} \cos\left(n \frac{2\pi t}{T}\right) = T \sum_{n} \delta(t - nT)$$

we have

$$\Delta = AT \cos qx \sum_{n} \delta(t - nT) \tag{6}$$

Equation (6) shows that atoms absorb detuning in short pulses with period $T$. The $n$th pulse time is $t_n = nT$; denote $z_n = z(t_n - 0)$, $\dot{z}_n = \dot{z}(t_n - 0)$. The pulses affect only the velocity of the linear oscillator. We have

$$\begin{aligned} z(t_n + 0) &= z(t_n - 0) = z_n \\ \dot{z}(t_n + 0) &= \dot{z}(t_n - 0) + \frac{q}{2} AT \sin qz \end{aligned} \tag{7}$$

From equations (3) and (7) we have

$$\begin{aligned} z_{n+1} &= \cos \omega T + \frac{1}{\omega} (\dot{z}_n + \frac{qAT}{2} \sin qz_n) \sin \omega T \\ z_{n+1} &= (\dot{z}_n + \frac{qAT}{2} \sin qz_n) \cos \omega T - \omega z_n \sin \omega T \end{aligned} \tag{8}$$

Let $k = q^2AT/2\omega$, $u = q\dot{z}/\omega$, $v = -qz$, and $\alpha = \omega T$; we rewrite equations (8) as

$$\begin{aligned} u_{n+1} &= (u_n + k \sin v_n) \cos \alpha + v_n \sin \alpha \\ v_{n+1} &= -(u_n + k \sin v_n) \sin \alpha + v_n \cos \alpha \end{aligned} \tag{9}$$

**2.** $\Delta$ expressed as the product of the probability distribution of the momentum and the kinetic energy of atoms. When $\Delta > 0$, atoms absorb $\Delta$, which is converted to the atoms' kinetic energy, whereas when $\Delta < 0$, atoms lose $-\Delta$ kinetic energy, which is converted to into the atoms' transition energy (Yang *et al.*, 1997). Therefore we can assume that detuning $\Delta$ is directly proportional to the product of the probability distribution of the momentum and kinetic energy of atoms,

$$\Delta = -Ae^{-\beta V^2}V^2 \sum_n \delta(t - nT) \tag{10}$$

where $V$ is the speed of translational motion and $\beta$ is a constant.

When $\beta$ is a small value

$$\Delta = AV^2 (\beta V^2 - 1) \sum_n \delta(t - nT) \tag{11}$$

Let $u = q\dot{z}/\omega$, $v = V = -qz$, and $\alpha = \omega T$; we obtain

$$\begin{aligned} u_{n+1} &= \left[ u_n + \frac{Aq^2}{\omega} (\beta v_n^3 - v_n) \right] \cos\alpha + v_n \sin\alpha \\ v_{n+1} &= -\left[ u_n + \frac{Aq^2}{\omega} (\beta v_n^3 - v_n) \right] \sin\alpha + v_n \cos\alpha \end{aligned} \tag{12}$$

## 3. TRAPPING OF ATOMS IN JC MODEL IN PHASE SPACE

We noticed that $u$ and $v$ are proportional to coordinate and velocity, respectively. Now we study the trapping of atoms in the JC model and their chaotic motion.

**1.** In our numerical calculation, the orders of maginitude of the parameters are

$$k = 0\text{–}10^2, \qquad u_0 = 10^{-2}\text{–}10^2, \qquad v_0 = 10^0\text{–}10^1, \qquad \alpha = 10^{-1}\text{–}10^{-3}$$

In order to show the trapping of the atom clearly, we treat (5) in phase space with position is scaled as with $v$. Then we expand the coordinate $v$; the trajectory of the oscillator is two ellipses, which correspond to two levels of the atom, respectively.

Figure 1 describes the process. In Fig. 1 the trajectory is a standard ellipse owing to the absence of a perturbation potential. In Fig. 1b with a perturbation coefficient $k = 0.40$, the ellipse trajectory is disturbed locally; some parts are concave toward the center, corresponding to low potential and high kinetic energy, other parts are convex toward the center, corresponding to high potential and low kinetic energy. At some high-potential points of a low-energy-level ellipse upward and downward orbits are very close to each other. When $k = 0.54$ (Fig. 1c) the oscillator trajectory becomes closed and trapping occurs. The situation is the same for the high-level ellipse (Figs. 1d, 1e); the oscillator trajectory drops into a potential trap. The initial position is not in the center of the potential trap, and moves

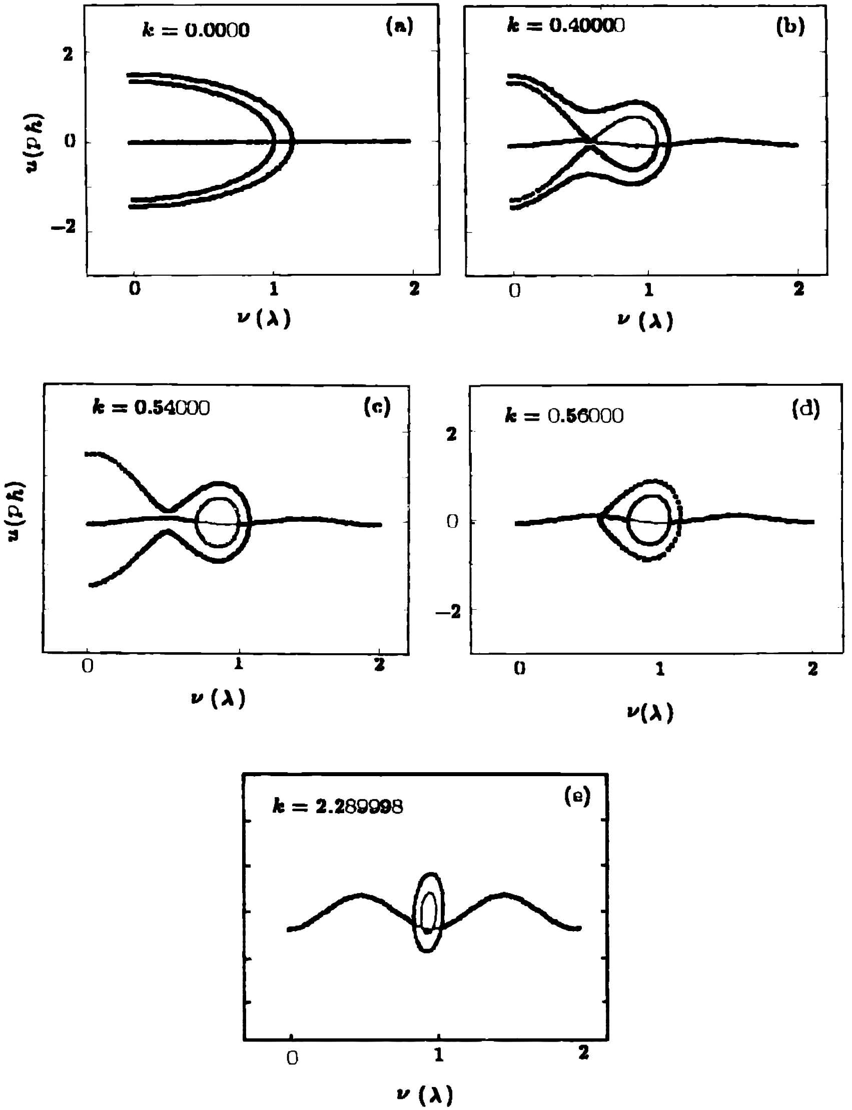


**Fig. 1.** The trapping of atoms in the JC model.

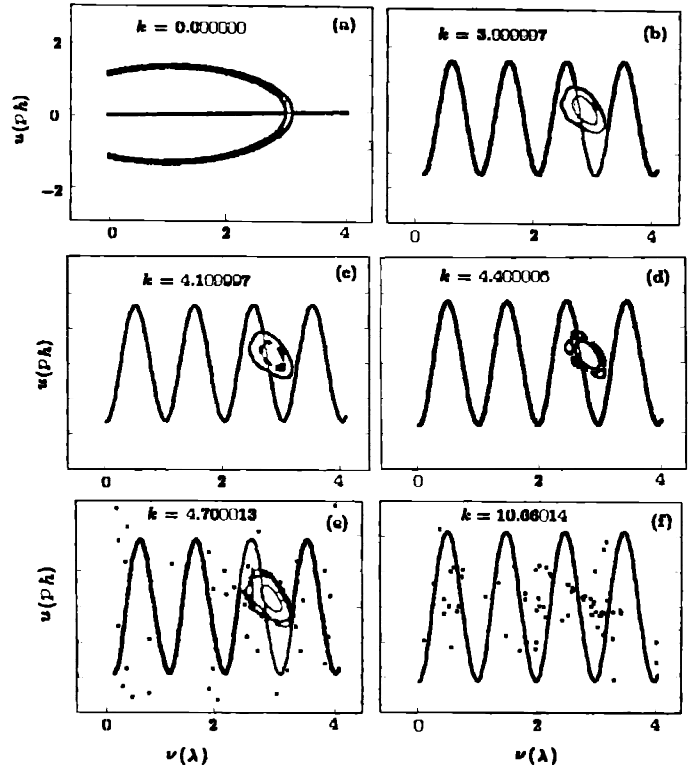


**Fig. 2.** Spectroscopic splitting of trapped atoms.

toward the center of the potential trap step by step. This gives the whole process of the trapping of atoms from the energy view of point.

**2.** Let us solve equation (12). We draw similar conclusions to equation (10) (Fig. 2). When the perturbation intensity is increased, an interesting phenomenon occurs, level splitting and complicated periodic motion. Finally, the trapping of atoms disappears and chaos occurs; the oscillator walks randomly in phase space (Fig. 2f). The JC model of a two-level system allows degeneration. Therefore the spectroscopic splitting of trapped atoms is reasonable. We draw the same conclusions from classical chaos theory and find the details of the dynamics. Trapping is limited by perturbation. When the perturbation intensity exceeds some critical value, chaos occurs and trapping is destroyed. We also find the avoidance of level, crossing, which is an important sign of quantum chaos (Fig. 3). Using this procedure, we can study quantum movement from regular to nonregular.

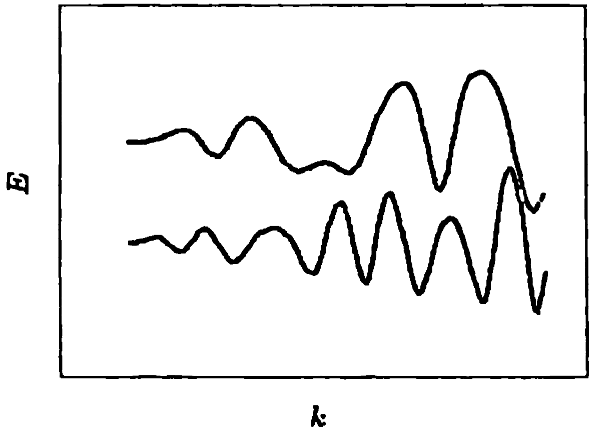


**Fig. 3.** The avoidance of level crossing.

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